Irrational germs.

We now focus on irrational germs

$f(z) = \lambda z(1+o(1))$, $\lambda = e^{2\pi i \alpha}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

The conjugacy ϕ is unique if we impose $\phi'(0)=1$.

Prop: any $f : (\mathbb{C},0) \circlearrowleft$ irrational germ is formally linearisable.

Proof: We want to show that the equation $\phi \circ f = \lambda \phi$ has a formal solution $\forall f$ as above

Write $f(z) = \lambda z(1+\varepsilon(z))$ $\quad 1+\varepsilon(z) = \sum \varepsilon_n z^n$.

$\phi(z) = z \sum_{n \geq 0} \phi_n z^n$ with $\phi_0 = \varepsilon_0 = 1$.

$$\phi \circ f = \cancel{\lambda z}(1+\varepsilon) \sum \lambda^k \phi_k (1+\varepsilon(z)) z^k = \cancel{\lambda z} \sum_k \lambda^k \phi_k z^k (1+\varepsilon(z))^{k+1} =: \sum I_n z^n$$

$$\lambda \phi = \cancel{\lambda z} \sum \phi_n z^n$$

Here $I_n = \sum_{\substack{k, H \in \mathbb{N}^{k+1} \\ k+|H|=n}} \lambda^k \phi_k \cdot \varepsilon_H$, with $\varepsilon_H = \varepsilon_{h_1} \cdots \varepsilon_{h_{k+1}}$

Notice that $I_n = \phi_n \cdot \lambda^n + R_n(\phi_0, \phi_1, \dots \phi_{n-1}, \varepsilon_0, \dots, \varepsilon_n)$, where R_n is a suitable polynomial. The conjugacy equation is then equivalent to the system of equations $\phi_n = \phi_n \lambda^n + R_n$, which has solution $\phi_n = -\dfrac{R_n}{1-\lambda^n}$. $(*_n)$

$(*_n)$ allows to solve the conjugacy equations recursively on n, since $\lambda^n \neq 1$ $\forall n$

Notice that if $\varepsilon_h = 0$ $\forall h > 0$, then $I_n = \lambda^n \cdot \phi_n$, and $(*_n)$ implies $\phi_n = 0$ $\forall n > 0$. $\square$

Rem: The same proof works also in the attracting/repelling cases, and holds over any field (even non-algebraically closed, and in any characteristic).

A natural question ~~is~~ is if whether f is also analytically linearisable, i.e., if the conjugacy ϕ defined recursively above is actually convergent.

We saw that the recursion is of the form: 6.34

$\phi_n = \dfrac{R_n}{\lambda^n - 1}$. In particular, it is natural to expect that the convergence of $\sum \phi_n z^n$ can depends on how ~~sm~~ small $|\lambda^n - 1|$ can be, thing that ~~…~~ depends on the arithmetic properties of α, and ~~…~~ in particular on how α can be well approximated by rational numbers.

This problem is known as "small denominators problem".

Def: An irrationally indifferent fixed point germ $f : (\mathbb{C}, 0) \circlearrowleft$ is called

- Siegel point if f is analytically linearizable
- Cremer point if f is not analytically linearizable.

Rem: This definition alludes to the fact that (from the analysis of possible dynamics on hyperbolic surfaces) if $f : \hat{\mathbb{C}} \circlearrowleft$ is a rational map with irrational fixed point z_0, then f_{z_0} is analytically linearizable $\Leftrightarrow$ $z_0 \in F(f)$, and the connected component of $F(f)$ containing z_0 is a Siegel disc.

Prop: $f : (\mathbb{C}, 0) \circlearrowleft$ irrational indifferent germ. Then the following are equivalent:

(1) f is analytically linearizable.
(2) f is topologically linearizable.
(3) the sequence $\{f^n\}$ is uniformly bounded in a neighborhood of 0.

Proof: $(1 \Rightarrow 2)$ is trivial, as is $(2 \Rightarrow 3)$

$(3 \Rightarrow 1)$: Assume there exist $M >> 0$, $\varepsilon << 1$ so that $|f^n(z)| \le M$ $\forall n$, $|z| \le \varepsilon$.

Set $\Phi_n(z) = \dfrac{1}{n} \sum_{j=0}^{n-1} \lambda^{-j} \cdot f^j(z)$.

$|\Phi_n(z)| \le M$ for $|z| \le \varepsilon$, ~~…~~ Hence (Φ_n) is a uniformly bounded sequence of analytic functions, so it admits a convergent subsequence. $\Phi_{n_k} \to \Phi_\infty$

Since $f'(0) = \lambda$, so $\Phi_n'(0) = 1$ so $\Phi_\infty'(0) = 1$.

Moreover $\Phi_n \circ f = \dfrac{1}{n} \sum_{j=0}^{n-1} \lambda^{-j} f^{j+1} = \lambda \Phi_n + \dfrac{\lambda}{n}(\lambda^{-n} f^n - id) \overset{\text{use the hyp.}}{=} \lambda \Phi_n + \mathcal{O}\left(\frac{1}{n}\right)$

Taking logarithms, we find that $(*)$ is equivalent to:

$q_{n+1} - q_n \leq C(\delta)(\cdot d^{2^{q_n}} - 1)$, where $C(\delta)$ is a constant depending only on δ ($C(\delta) \simeq \log \frac{1}{\delta} \to \infty$ when $\delta \to 0$.)

If we take (q_n) so that $q_{n+1} - q_n$ grows very rapidly, for example: $q_{n+1} \overset{(?)}{=} q_n + e^{n 2^{q_n}}$, then $(*)$ is eventually violated for any $\delta > 0$, $d \in \mathbb{N} \setminus \{0,1\}$.

□

Corollary: $\mathbb{R}/\mathbb{Z} \setminus \mathcal{L}$ is dense in $\mathbb{R} \setminus \mathbb{Z}$.

Proof: Any rational number ~~in~~ in $[0,1)$ can be written in base two as $0, \varepsilon_1 \varepsilon_2 \varepsilon_3 \dots \varepsilon_k$ with $\varepsilon_j \in \{0,1\}$ and $k \in \mathbb{N}$.

In particular it can be written as $\sum_{j=1}^{r} 2^{-q_j} = \alpha_0$ for some increasing sequence $q_j \in \mathbb{N}$.

We then pick $q_{r+1}, q_{r+2} \dots$ growing rapidly as in the proof of the previous theorem, and we have $\alpha = \alpha_0 + \sum_{j=r+1}^{\infty} 2^{-q_j}$ close as we want to α_0, $\alpha \notin \mathcal{L}$. □

We will show now that $\mathcal{L} \neq \emptyset$, and in fact $\mathcal{L}$ is dense and of full measure on $\mathbb{R}/\mathbb{Z}$.

Historic: The theorem above is Pfeiffer 1917.

Cremer (1938): if $\liminf |\lambda^n - 1|^{\frac{1}{n}} = 0 \Rightarrow \alpha \notin \mathcal{L}$.

Siegel (1942): first example of $\alpha \in \mathcal{L}$.

(For this part see [Carleson - Gamelin])

Hence the limit Φ_∞ satisfies $\Phi_\infty \circ f = \lambda \Phi_\infty$. $\square$

A natural question is ~~[illegible]~~ if the above condition of linearisability is always satisfied. We say that f is linearisable if it is analytically, or equivalently topologically, linearisable.

We denote by $\mathcal{L} \subset \mathbb{R}/\mathbb{Z}$ the set of $\alpha \in \mathbb{R}/\mathbb{Z}$ so that $\forall f : (\mathbb{C},0)\circlearrowleft$, $f'(0) = \lambda = e^{2\pi i \alpha}$ are (analytically) linearisable.

First, we show that $\mathcal{L} \neq \mathbb{R}/\mathbb{Z}$.

Theorem (Pfeiffer): There exist α so that any polynomial $f \in z\mathbb{C}[z]$ with $f'(0) = e^{2\pi i \alpha}$, $\deg f = d \geq 2$ is not (analytically) linearisable

Proof: Let $f(z) = z^d + \dots + \lambda z$ (up to linear conjugacy, the coeff of z^d is 1)

And suppose there exists ϕ defined on a disc $D(0,\delta)$ $\delta > 0$, so that $\phi \circ f = \lambda \phi$.

f^n has d^n fixed point, satisfying the equation:

$$f^n(z) - z = z^{d^n} + \dots + \underset{\neq 0}{(\lambda^n - 1)} z = 0.$$

~~One root~~ There is one (simple) root at 0. Call the others $z_1^{(n)} \dots z_{d^n-1}^{(n)}$.

Notice that $f^n(z) = \phi^{-1}(\lambda^n \phi(z))$ has no fixed points besides 0 in $D(0,\delta)$, and $|z_j^{(n)}| \geq \delta$ $\forall j$.

It follows that $\delta^{d^n - 1} \leq \prod_{j=1}^{d^n-1} |z_j^{(n)}| = |\lambda^n - 1|$. $(*)$

We will now construct λ not verifying $(*)$.

Let $q_1 < q_2 < \dots$ be a strictly increasing sequence of positive integers, and set:

$\alpha = \sum_{j=1}^{\infty} 2^{-q_j}$, $\lambda = e^{2\pi i \alpha}$. For $k >> 0$, we have that

$$|\lambda^{2^{q_n}} - 1| \sim \frac{1}{2^{q_{n+1}-q_n}} \quad \left(\lambda^{2^q} = e^{2\pi i \sum_{j>n} 2^{q_n - q_j}} \Rightarrow |\lambda^{2^{q_n}} - 1| \sim \sum_{j>n} 2^{q_n - q_j} \sim 2^{q_n - q_{n+1}} \right)$$

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Theorem (Yoccoz). $\alpha \in \mathcal{L}$ if (and only if) $P_\alpha(z) = e^{2\pi i\alpha} z + z^2$ is analytically linearisable [Proof: Buff - Hubbard]

Proof (idea) let $f(z) = \lambda z + a_0 z^2 + u(z)$ $\quad u(z) = o(z^2)$ be any germ.

By conjugating with a linear map if necessary, we may assume f is holomorphic on $\mathbb{D}$. Define: $g_b : D(0, \frac{1}{|b|}) \to \mathbb{C}$ and $f_a : \mathbb{D} \to \mathbb{C}$ by:

$$z \mapsto \lambda z + z^2 + \frac{1}{b} u(bz) \qquad z \mapsto \lambda z + az^2 + u(z) = \frac{1}{a} g_{\frac{1}{a}}(az)$$

In particular $f_a \simeq g_{\frac{1}{a}}$.

When $b \to 0$, the domain of definition of g_b grows to cover any given compact $K \subset \mathbb{C}$, and $g_b \to P_\alpha$ uniformly on such compacts.

Claim 1: If P_α is linearisable, then $\exists r_0 > 0$, $r_1 > 0$ such that $\forall b$, $|b| \leq r_0$, then g_b is linearisable, and $D(0, r_1)$ is contained in the Siegel disc of g_b.

Rem: read on f_a, it says that $\forall a$, $|a| \geq \frac{1}{r_0}$, f_a is linearisable and $D_{\frac{r_1}{|a|}}$ is contained in the Siegel disk of f_a.

If $|a_0| \geq \frac{1}{r_0}$, we are done.

$D_r = D(0, r)$

If not: **Claim 2**: For any $z \in D_{r_0 r_1}$, any $a \in D_{\frac{1}{r_0}}$, any $n \geq 0$, the iterate $f_a^n(z)$ is well defined and belongs to $\mathbb{D}$.

From claim 2: we deduce that $|f_a^n(z)| < 1$ $\forall n$, $|z| < \frac{1}{r_0}$, and f_a is linearisable by the caracterisation given above. $\square$

Proof of Claim 2: Fix z, $|z| < r_0 r_1$, and prove by induction on n that $a \mapsto f_a^{\circ n}(z)$ is holomorphic on $D_{\frac{1}{r_0}}$, and takes values in $\mathbb{D}$.

For $n = 0$ $\quad a \mapsto z$ clearly satisfies the condition.

Assume it is true for $n-1$.

Then, $z \mapsto f_a^n(z) = \lambda f_a^{n-1}(z) + a\left(f_a^{n-1}(z)\right)^2 + u\left(f_a^{n-1}(z)\right)$

is well defined and holomorphic (u is defined on $\mathbb{D} \ni f_a^{n-1}(z)$).

By maximum modulus principle, $a \mapsto |f_a^n(z)|$ takes its maximum when $|a| = \frac{1}{r_0}$.

But by Claim 1, (f_a is linearizable and) D_{2r_1} is contained in the Siegel disc of f_a. $\Rightarrow |f_a^n(z)| < 1$ and we are done $\square$

(IDEA)

Proof claim 1: Step 1: $P_\alpha^{-1}(D_4) \subset D_3 \leadsto P_\alpha : P_\alpha^{-1}(D_4) \to D_4$ is a proper mapping of degree 2.

For $|b| << 1$ (say $< \varepsilon$), the same holds for $g_b : U_b = g_b^{-1}(D_4) \cap D_4 \to D_4$.

$\subset\subset D_4$

(called quadratic polynomial like mapping).

Riemann-Hurwitz: $\exists!$ critical point $w_b \in U_b$ for g_b, and $b \mapsto w_b$ is holomorphic (by implicit function theorem)

- Step 2: $|b| < \varepsilon \leadsto \forall n \geq 0$, $g_b^n(w_b)$ is well defined and belongs to $\overline{U_b}$.

Either $g_b \simeq z \mapsto \lambda z$ and the boundary of the Siegel disk is contained in $\overline{\{g_b^n(w_b)\}}$

or $g_b \not\simeq z \mapsto \lambda z$, and 0 is a non-isolated point in $\overline{\{g_b^n(w_b)\}}$.

- Step 3: If P_α is linearizable and $D_{2r_1} \subset$ Siegel disk of P_α.

If $|b| < r_0 := \frac{\varepsilon \cdot r_1}{16}$, then g_b is linearizable and $D_{r_1} \subset$ Siegel disk of g_b